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Author for correspondence:
L. Vergori
e-mail: luigi.vergori@unipg.it

A constitutive model for transversely isotropic dispersive materials

A. Amendola[1], J. de Castro Motta[1], G. Saccomandi[2,3,4] and L. Vergori[2]

[1]Dipartimento di Ingegneria Civile, Università di Salerno, via Giovanni Paolo II 132, 84084 Fisciano (SA), Italy
[2]Dipartimento di Ingegneria, Università degli Studi di Perugia, via Goffredo Duranti 93, 06125 Perugia, Italy
[3]School of Mathematics, Statistics and Applied Mathematics, NUI Galway, University Road, Galway, Ireland
[4]Sezione INFN dell'Università degli Studi di Perugia, via Alessandro Pascoli 23c, 06125 Perugia, Italy

AA, 0000-0002-2562-881X; GS, 0000-0001-7987-8892; LV, 0000-0002-0332-5367

Due to their intrinsic complexity, it is not easy to model the mechanical behaviour of biomaterials. Despite the challenges they faced, some researchers have presented mathematical models to describe some aspects of the mechanical response of soft tissues. Since most of the materials of biological interest are composites made of different constituents reinforced by collagen and/or elastin fibres, material dispersion and anisotropy are non-negligible. Within the theory of elasticity, several models for anisotropic materials have been developed. The same cannot be said for anisotropic dispersive materials. The models available in the literature that account for material dispersion are valid only for isotropic materials. This paper aims at introducing a very general model for anisotropic dispersion in transversely isotropic bodies within the theory of simple materials of differential type. Our model has the potential to represent a first step towards a better understanding of the mechanical response of fibre-reinforced soft materials.

1. Introduction

Within the classical formulation of the theory of elasticity, bulk waves propagate without dispersion,

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and dispersive effects are exclusively due to boundary effects. These boundaries may be macroscopic (e.g. the free lateral surfaces of an elastic rod), mesoscopic (e.g. the interfaces between two constituents in a composite material, grain boundaries, pores) or microscopic (with the same order of magnitude as the characteristic length of the molecular lattice and/or network as in the case of dislocations). In the last three decades, there has been a considerable effort in applying the theory of elasticity to soft materials [1]. In particular, several nonlinear mathematical models that account for the reinforcement of soft tissues by collagen and/or elastin fibres have been developed. Many of them have been used to describe real phenomena. However, despite their sophisticatedness, these models are far from satisfactorily characterizing the real mechanical behaviour of soft tissues as the tremendous complexity of biomaterials makes any idealization of theirs as elastic solids a mere oversimplification.

Arteries are soft tissues reinforced by collagen fibres and their mathematical modelling has attracted the attention of many scholars. A detailed review on the development of the constitutive models for the mechanical response of arteries can be found in [2]. Myneni & Rajagopal [2] have also provided a model that accounts also for the viscoelasticity, anisotropy and compressibility of arteries. An aspect that has been completely ignored by the authors is the presence of an inherent material characteristic length in arteries. Like any structured material, real biomaterials possess inherent material characteristic lengths. (Besides arteries, any biomaterial is usually structured and another clear example is provided by the human skin as it is a composite consisting of strong fibrils and fluid-like matrix components.)

In analogy with the theories of elastic rods and continua with microstructure, it is then reasonable to expect that the boundary conditions at the interfaces between two different constituents and/or the microstructure originate dispersive effects also in biomaterials. Despite this, dispersive phenomena have been rarely taken into account when modelling the constitutive behaviour of soft tissues, even though some measurements of the shear wave dispersion in tendons [3], haemoglobin [4] and pregnant cervix [5] are available, and modern advances in diagnostic techniques based on wave propagation (such as elastography) suggest that material dispersion may be used as a new index for the mechanical characterization of soft tissues by ultrasound [6].

A theory of material dispersion due to an inherent characteristic length should bridge the mesoscopic structure of a given body with its macroscopic mechanical behaviour. For crystalline materials, this can be done by considering the quasi-continuum limit of lattice dynamics [7–9]. This approach is however nearly impassable for amorphous materials with a polymeric network [10].

For biomaterials, a pure phenomenological theory of material dispersion could be based on the theory of continua with microstructure [11], or second gradient bodies [12], or more general non-local theories [13]. All these theories do not fall under the rigorous and reassuring field of simple materials *à la* Truesdell & Noll [14]. In particular, within these theories, additional boundary conditions are necessary (some of them without a clear physical meaning) and may lead to ill-posed problems. As an example, the Boussinesq equation (which represents a generalization of the classical wave equation within the theory of non-local elasticity) is ill-posed as small amplitude solutions with short wavelengths are unstable [15].

In [16] Rubin, Rosenau and Gottlieb (hereinafter RRG for brevity) developed an original approach for modelling dispersion caused by an inherent characteristic length. The RRG model can be viewed as the phenomenological version of the discrete theory proposed by Rosenau [8], falls within the theory of simple materials [17], does not require the introduction of additional initial and/or boundary conditions, and can be easily applied to amorphous materials. It has been used also to study the propagation of finite-amplitude bulk waves in isotropic dispersive solids [18,19], and hence to generalize prior studies in crystalline materials [20,21] and amorphous solids [22]. Incidentally, within the RRG theory terms containing higher-order mixed derivatives with respect to spatial and temporal variables have to be added to the classical wave equation producing the nonlinear Love equation. This equation does not need additional boundary conditions, is well-posed and its small amplitude solutions are stable at all wavelengths [23].

The RRG model has been derived only for isotropic materials. A generalization of this model is then necessary when modelling dispersive anisotropic materials. This paper is aimed at proposing a phenomenological continuum model for transversely isotropic dispersive materials. The model that will be developed in the next sections represents a first step toward a more satisfactory description of the mechanical response of anisotropic soft materials of biological interest such as tendons, organs, blood vessels etc. These composite materials consists of an elastic ground matrix reinforced by elastin and/or collagen fibres. Therefore, it is natural to expect that the boundary effects between the various constituents generate dispersive phenomena. In particular, for soft tissues with a single family of reinforcing fibres, one expects three characteristic lengths: one for the family of fibres, and the other two for the elastic ground matrix in directions parallel and normal to the fibres. Here, we focus our attention on this class of materials.

The structure of the paper is as follows. In the next section, we write down the basic equations and we review the RRG model by proposing a different and, in our opinion, simpler derivation. Within the theory of simple materials of differential type, we split the active part of the Cauchy stress tensor into the sum of two contributions: one due to the elasticity of the material, the other due to material dispersion. Based on a series of results by Edelen [24–26] on the form of constitutive relations consistent with the laws of thermodynamics, and assuming that dissipative effects are negligible, we model the dispersive part of the Cauchy stress as a tensor field whose power per unit volume equals the rate of change of a contribution to the density of internal energy. In other words, we limit our analysis to phenomena in which energy is conserved. This approach can be easily generalized to anisotropic materials. In particular, we adapt the analysis performed in the isotropic case to account for anisotropic material dispersion in a transversely isotropic elastic solid (§3). We then derive the linearized theory (which is appropriate at infinitesimal strains and strain rates) by which we discern three characteristic lengths, and discuss the propagation of anti-plane shear waves (§4). In §5, we study the propagation of solitary waves at finite strains and observe that in some particular cases, pulse waves (i.e. solitary waves with compact support) can propagate. We then conclude with some remarks.

2. Basic equations

Let $\boldsymbol{X} = X_1\boldsymbol{e}_1 + X_2\boldsymbol{e}_2 + X_3\boldsymbol{e}_3$ be the position vector (relative to an origin O) of a particle P of a body $\mathscr{B}$ at the initial time $t = 0$, and $\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3$ be the position vector (relative to the same origin O) of the same particle at time t. For convenience, the configuration $\mathcal{B}_r$ occupied by the body $\mathscr{B}$ at the initial time is taken as the reference configuration. A motion of $\mathscr{B}$ in the time interval $[0, T)$ is a mapping $\boldsymbol{\chi}$ defined in $\mathcal{B}_r \times [0, T)$ such that, for any $t \in [0, T)$, $\boldsymbol{\chi}_t \equiv \boldsymbol{\chi}(\cdot, t)$ is one-to-one, and $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t)$. Since we are here interested in smooth motions, we assume that $\boldsymbol{\chi} \in \mathrm{C}^\infty(\mathcal{B}_r \times [0, T))$.

The configuration $\mathcal{B}_t = \boldsymbol{\chi}_t(\mathcal{B}_r) = \boldsymbol{\chi}(\mathcal{B}_r, t)$ occupied by the body at time t is called the current configuration. The deformation gradient $\boldsymbol{F}$ is the second-order tensor field defined as $\boldsymbol{F} = \partial\boldsymbol{x}/\partial\boldsymbol{X}$, in terms of which the right and left Cauchy–Green deformation tensors $\boldsymbol{C}$ and $\boldsymbol{B}$ and the velocity gradient $\boldsymbol{L} = \partial\boldsymbol{v}/\partial\boldsymbol{x}$, with $\boldsymbol{v}$ being the spatial description of the velocity field, read

$$\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F},\ \boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T \quad \text{and} \quad \boldsymbol{L} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1}. \tag{2.1}$$

Henceforth, the superposed dot and, occasionally to avoid confusion, the differential operator $\mathrm{D}/\mathrm{D}t$ denote the material time derivative.

The symmetric part of the velocity gradient $\boldsymbol{D} = (\boldsymbol{L} + \boldsymbol{L}^T)/2$ represents the strain-rate tensor and the skew-symmetric part $\boldsymbol{W} = (\boldsymbol{L} - \boldsymbol{L}^T)/2$ is the spin tensor. Given $n \in \mathbb{N}$, the nth Rivlin–Ericksen tensor is defined as

$$\boldsymbol{A}_n = \boldsymbol{F}^{-T}\frac{\mathrm{D}^n\boldsymbol{C}}{\mathrm{D}t^n}\boldsymbol{F}^{-1}. \tag{2.2}$$

It is easy to check that

$$\boldsymbol{A}_1 = 2\boldsymbol{D}, \tag{2.3a}$$

and, for any $n \in \mathbb{N}$, the $(n+1)$th Rivlin–Ericksen tensor can be computed through the recursive formula

$$\boldsymbol{A}_{n+1} = \dot{\boldsymbol{A}}_n + \boldsymbol{L}^T\boldsymbol{A}_n + \boldsymbol{A}_n\boldsymbol{L} = \underbrace{\dot{\boldsymbol{A}}_n - \boldsymbol{W}\boldsymbol{A}_n + \boldsymbol{A}_n\boldsymbol{W}}_{=\overset{\circ}{\boldsymbol{A}}_n} + \boldsymbol{D}\boldsymbol{A}_n + \boldsymbol{A}_n\boldsymbol{D}$$

$$= \overset{\circ}{\boldsymbol{A}}_n + \boldsymbol{D}\boldsymbol{A}_n + \boldsymbol{A}_n\boldsymbol{D}, \tag{2.3b}$$

where the superposed circle denotes the co-rotational time derivative. From (2.3*a*) to (2.3*b*), one deduces that the Rivlin–Ericksen tensors (2.2) are objective.

In this paper, we shall introduce constitutive relations for the stress tensor that depend only on the first two Rivlin–Ericksen tensors. From (2.3*a*) to (2.3*b*), it is easy to deduce that these two tensors are related through

$$\boldsymbol{A}_2 = \overset{\circ}{\boldsymbol{A}}_1 + \boldsymbol{A}_1^2. \tag{2.4}$$

Our subsequent analysis holds for bodies made of an incompressible homogeneous material. These bodies can undergo only isochoric motions for which $\det \boldsymbol{F} = 1$ or, equivalently for smooth motions,

$$\operatorname{div}\boldsymbol{v} = \operatorname{tr}\boldsymbol{D} = 0. \tag{2.5}$$

As a direct consequence of the incompressibility and homogeneity of the body, the mass density ρ is constant, and, given (2.4) and (2.5), the first two Rivlin–Ericksen tensors satisfy the relation

$$\operatorname{tr}\boldsymbol{A}_2 = \operatorname{tr}\boldsymbol{A}_1^2. \tag{2.6}$$

Assuming valid Cauchy's postulate on the internal actions in a continuum, denoting $\boldsymbol{a}$ the spatial description of the acceleration field and $\boldsymbol{T}$ the Cauchy stress tensor, in the absence of body forces, from the equations of balance of linear and angular momenta one derives the equation of motion

$$\rho\boldsymbol{a} = \operatorname{div}\boldsymbol{T}, \tag{2.7}$$

and deduces that $\boldsymbol{T}$ is symmetric. The equation of motion (2.7) is in Eulerian form. In what follows, since we are interested in dispersive elastic solids, we find it more convenient to write the equations of motion in Lagrangian form in which the kinematic fields and the stress tensor are expressed as functions of the particle $\boldsymbol{X}$ and time t. Then, introducing the first Piola–Kirchhoff stress tensor $\boldsymbol{P} = \boldsymbol{T}\boldsymbol{F}^{-T}$, the equation of motion in Lagrangian form reads

$$\rho\ddot{\boldsymbol{x}} = \operatorname{Div}\boldsymbol{P}, \tag{2.8}$$

where $\ddot{\boldsymbol{x}}$ is the material description of the acceleration field and Div is the material divergence (defined in terms of the derivatives with respect to the material coordinates X_1, X_2 and X_3).

The Piola–Kirchhoff stress tensor is not symmetric. However, it satisfies the identity $\boldsymbol{P}\boldsymbol{F}^T = \boldsymbol{F}\boldsymbol{P}^T$ as an immediate consequence of its definition and the symmetry of the Cauchy stress tensor.

(a) Material dispersion in isotropic materials

To account for the effects due to dispersion within the theory of incompressible simple materials of differential type, and guided by previous studies [16,17], we take the Cauchy stress tensor as the sum of three independent contributions: the isotropic tensor

$$\boldsymbol{T}_C = -p\boldsymbol{I} \tag{2.9}$$

associated with the internal constraint of incompressibility and with the indeterminate Lagrange multiplier p having the same physical dimensions as the pressure, an elastic part $\boldsymbol{T}_E = \hat{\boldsymbol{T}}_E(\boldsymbol{B})$ and

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a term in the form $\boldsymbol{T}_D = \hat{\boldsymbol{T}}_D(\boldsymbol{A}_1, \boldsymbol{A}_2)$ that models material dispersion. In other words, we assume that

$$\boldsymbol{T} = -p\boldsymbol{I} + \boldsymbol{T}_E + \boldsymbol{T}_D. \tag{2.10}$$

For an incompressible isotropic hyperelastic solid, the stress tensor $\boldsymbol{T}_E$ is determined from a strain energy density (per unit volume) $W = \hat{W}(I_1, I_2)$, where since $\det \boldsymbol{B} = 1$,

$$I_1 = \mathrm{tr}\boldsymbol{B} \quad \text{and} \quad I_2 = \mathrm{tr}\boldsymbol{B}^{-1} \tag{2.11}$$

are the first two principal invariants of the left Cauchy–Green deformation tensor. More precisely, denoting $\hat{W}_i$ $(i = 1, 2)$ the derivative of the strain energy density with respect to the invariant I_i, the active part of the stress tensor accounting for the elasticity of the body is given by the following constitutive relation:

$$\boldsymbol{T}_E = \hat{\boldsymbol{T}}_E(\boldsymbol{B}) = 2\hat{W}_1\boldsymbol{B} - 2\hat{W}_2\boldsymbol{B}^{-1}. \tag{2.12}$$

In the early 1970s Edelen [24,25] investigated thoroughly the form of the constitutive relations consistent with the laws of thermodynamics. Specifically, on using his own words, Edelen proved that

> constitutive relations take the form of a sum of two collections of terms; the first collection being the quasistatic part of classic thermodynamics, and the second is the dynamic part. This dynamic part consists of a gradient of a dissipation potential with respect to the dynamic variables and a collection of terms which gives an identically zero supply of entropy.

Going back to the model for the Cauchy stress tensor (2.10), we note that the constraint stress (2.9) is powerless due to (2.5), while the elastic part (2.12) produces a zero supply of entropy as its power equals the rate of change of the elastic potential W. Indeed, from the Cayley–Hamilton theorem, (2.1) and (2.5), we deduce that

$$\boldsymbol{T}_E \cdot \boldsymbol{D} = (2\hat{W}_1\boldsymbol{B} - 2\hat{W}_2\boldsymbol{B}^{-1}) \cdot \boldsymbol{L} = 2[\hat{W}_1\boldsymbol{F} + \hat{W}_2(I_1\boldsymbol{F} - \boldsymbol{B}\boldsymbol{F})] \cdot \dot{\boldsymbol{F}} = \hat{W}_1\dot{I}_1 + \hat{W}_2\dot{I}_2 = \dot{W}. \tag{2.13}$$

Edelen [26] proved also that, in modelling the stress tensor of incompressible materials, the constraint stress (2.9) is the sole powerless contribution. Therefore, limiting our attention to dispersive materials in which dissipative effects are negligible, in the light of Edelen's results, the tensor accounting for material dispersion is not powerless, is not derived from a dissipation potential and does not contribute to the supply of entropy. Then, like for the elastic part of the stress, the power per unit volume of $\boldsymbol{T}_D$ has to equal the rate of change of a contribution (say e_D) to the density (per unit mass) of internal energy, namely

$$\rho\dot{e}_D = \boldsymbol{T}_D \cdot \boldsymbol{D}. \tag{2.14}$$

To determine the most general constitutive relation $\boldsymbol{T}_D = \hat{\boldsymbol{T}}_D(\boldsymbol{A}_1, \boldsymbol{A}_2)$ (and hence to derive the specific energy e_D), we follow similar arguments as in [17] and take $\boldsymbol{T}_D$ of the same form as the Cauchy stress tensor of a second-grade fluid:

$$\boldsymbol{T}_D = \rho[\alpha_0(\delta_1)\boldsymbol{A}_1 + \alpha_1(\delta_1)\boldsymbol{A}_2 + \alpha_2(\delta_1)\boldsymbol{A}_1^2], \tag{2.15}$$

where, in view of the incompressibility and (2.6), the response functions α's depend only on the invariant $\delta_1 = \mathrm{tr}\boldsymbol{A}_1^2$. Then, we look for the restrictions on (2.15) for the stress power per unit volume $\boldsymbol{T}_D \cdot \boldsymbol{D}$ to satisfy (2.14). In so doing, from (2.4), we deduce that

$$\begin{aligned}\frac{\boldsymbol{T}_D \cdot \boldsymbol{D}}{\rho} &= \frac{\alpha_0(\delta_1)}{2}\mathrm{tr}\boldsymbol{A}_1^2 + \frac{\alpha_1(\delta_1)}{2}\overset{\circ}{\boldsymbol{A}}_1 \cdot \boldsymbol{A}_1 + \frac{1}{2}[\alpha_1(\delta_1) + \alpha_2(\delta_1)]\mathrm{tr}\boldsymbol{A}_1^3 \\ &= \frac{\alpha_1(\delta_1)}{4}\dot{\delta}_1 + \frac{\alpha_0(\delta_1)}{2}\delta_1 + \frac{1}{2}[\alpha_1(\delta_1) + \alpha_2(\delta_1)]\mathrm{tr}\boldsymbol{A}_1^3,\end{aligned} \tag{2.16}$$

whence, if $\alpha_0 = 0$ and $\alpha_1 + \alpha_2 = 0$, then

$$\boldsymbol{T}_D \cdot \boldsymbol{D} = \rho \frac{\alpha_1(\delta_1)}{4} \dot{\delta}_1 = \rho \frac{\mathrm{D}}{\mathrm{D}t} \left(\int \frac{\alpha_1(\delta_1)}{4} \, \mathrm{d}\delta_1 \right), \tag{2.17}$$

which implies that the contribution to the internal energy due to material dispersion is in the form

$$e_D = \hat{e}_D(\delta_1) = \int \frac{\alpha_1(\delta_1)}{4} \, \mathrm{d}\delta_1. \tag{2.18}$$

Note that the material parameter α_1 has physical dimension $[L^2]$, and, due to thermodynamic restrictions [27], it has to satisfy the requirement $\alpha_1(0) > 0$ to guarantee that $\hat{e}_D$ attains a minimum at equilibrium. The quantity $\sqrt{\alpha_1(0)}$ represents the material characteristic length.

We can conclude that the most general model for the stress tensor that accounts for material dispersion in incompressible isotropic materials and that depends on the first two Rivlin–Ericksen tensors reads

$$\boldsymbol{T}_D = \hat{\boldsymbol{T}}_D(\boldsymbol{A}_1, \boldsymbol{A}_2) = \rho \alpha_1(\delta_1)(\boldsymbol{A}_2 - \boldsymbol{A}_1^2). \tag{2.19}$$

This constitutive relation coincides with the model for $\boldsymbol{T}_D$ proposed in [16]. The derivation of (2.19) presented here is different from the one in [16] and can be easily generalized to anisotropic materials.

3. Material dispersion in transversely isotropic materials

We now adapt appropriately the analysis performed in the previous section to anisotropic materials and propose a constitutive model for material dispersion in transversely isotropic bodies. In particular, we derive a model for the stress tensor and the contribution to the internal energy density associated with material dispersion in a body with a preferred direction.

As before, we start by taking the Cauchy stress tensor as the sum of three independent terms: the constraint stress (2.9), the elastic part $\boldsymbol{T}_E$ and the contribution $\boldsymbol{T}_D$ due to material dispersion. Differently, from the isotropic case, for a transversely isotropic body both the elastic part and the term accounting for material dispersion must depend on a preferred direction.

Denoted $\boldsymbol{M}$, with $|\boldsymbol{M}| = 1$, the preferred direction in the reference configuration, for an incompressible transversely isotropic hyperelastic material the elastic part $\boldsymbol{T}_E$ is now derived from a strain energy density in the form $W = \tilde{W}(I_1, I_2, I_4, I_5)$, with I_1 and I_2 as in (2.11),

$$I_4 = \boldsymbol{C}\boldsymbol{M} \cdot \boldsymbol{M} = \boldsymbol{F}\boldsymbol{M} \cdot \boldsymbol{F}\boldsymbol{M} \equiv \boldsymbol{m} \cdot \boldsymbol{m} \quad \text{and} \quad I_5 = \boldsymbol{C}^2\boldsymbol{M} \cdot \boldsymbol{M}. \tag{3.1}$$

Specifically, $\boldsymbol{T}_E$ is given by

$$\boldsymbol{T}_E = 2\tilde{W}_1 \boldsymbol{B} - 2\tilde{W}_2 \boldsymbol{B}^{-1} + 2\tilde{W}_4 \boldsymbol{m} \otimes \boldsymbol{m} + 2\tilde{W}_5(\boldsymbol{m} \otimes \boldsymbol{B}\boldsymbol{m} + \boldsymbol{B}\boldsymbol{m} \otimes \boldsymbol{m}), \tag{3.2}$$

where, as before, the subscript i $(i = 1, 2, 4, 5)$ denotes differentiation with respect to I_i. Since any strain energy density can be defined up to an additive constant, we can take an additive constant such that the elastic potential energy vanishes in the reference configuration (for which $(I_1, I_2, I_4, I_5) = (3, 3, 1, 1)$), whence

$$\tilde{W}(3, 3, 1, 1) = 0. \tag{3.3}$$

On the other hand, on assuming that the reference configuration is stress-free one has that

$$2\tilde{W}_1(3, 3, 1, 1) + 4\tilde{W}_2(3, 3, 1, 1) = p_R \quad \text{and} \quad \tilde{W}_4(3, 3, 1, 1) + 2\tilde{W}_5(3, 3, 1, 1) = 0, \tag{3.4}$$

where p_R is the value of the Lagrange multiplier p in the reference configuration [28].

As in the previous section, bearing in mind the results by Edelen [24–26] and assuming that dissipative effects are negligible, the part of the stress tensor that accounts for material dispersion must satisfy a relation formally identical to (2.14), but with the energy density e_D depending

somehow on the preferred direction. To determine constitutive models for the stress tensor and the specific energy density in dispersive transversely isotropic materials, we take $\boldsymbol{T}_D$ of the same form as the Cauchy stress tensor for a transversely isotropic second-grade fluid [29]:

$$\boldsymbol{T}_D = \rho\Big[\phi_0 \boldsymbol{m}\otimes\boldsymbol{m} + \phi_1\boldsymbol{A}_1 + \phi_2\boldsymbol{A}_2 + \phi_3\boldsymbol{A}_1^2 + \phi_4(\boldsymbol{A}_1\boldsymbol{m}\otimes\boldsymbol{m} + \boldsymbol{m}\otimes\boldsymbol{A}_1\boldsymbol{m}) + \phi_5(\boldsymbol{A}_2\boldsymbol{m}\otimes\boldsymbol{m} + \boldsymbol{m}\otimes\boldsymbol{A}_2\boldsymbol{m}) + \phi_6(\boldsymbol{A}_1^2\boldsymbol{m}\otimes\boldsymbol{m} + \boldsymbol{m}\otimes\boldsymbol{A}_1^2\boldsymbol{m})\Big], \tag{3.5}$$

with ϕ_0 to ϕ_6 being functions of the invariants

$$I_4, \quad \delta_1 = \mathrm{tr}\boldsymbol{A}_1^2, \quad \delta_2 = \boldsymbol{A}_1\boldsymbol{m}\cdot\boldsymbol{m}, \quad \delta_3 = \boldsymbol{A}_2\boldsymbol{m}\cdot\boldsymbol{m}, \quad \delta_4 = \boldsymbol{A}_1^2\boldsymbol{m}\cdot\boldsymbol{m}, \tag{3.6}$$

and then find the restrictions on the response functions ϕ's for (2.14) to be satisfied.

As for the isotropic case, $\boldsymbol{T}_E$ and $\boldsymbol{T}_D$ are independent in the sense that the stress tensor accounting for material dispersion does not affect the elastic part of the stress. Then, since the invariant I_4 occurs in both models for $\boldsymbol{T}_E$ and $\boldsymbol{T}_D$, to guarantee the independence of the two contributions, we require that $\boldsymbol{T}_D$ vanishes at equilibrium. For this reason, we require that the response function ϕ_0 satisfies the restriction

$$\phi_0(I_4, 0, 0, 0, 0) = 0, \tag{3.7}$$

for any $I_4 > 0$.

On the other hand, from (2.1), (2.4) and the definition of $\boldsymbol{m}$, we deduce that

$$\dot{\boldsymbol{m}} = \boldsymbol{L}\boldsymbol{m}, \tag{3.8a}$$

$$\dot{I}_4 = \boldsymbol{A}_1\boldsymbol{m}\cdot\boldsymbol{m}, \tag{3.8b}$$

$$\dot{\delta}_2 = \delta_3, \tag{3.8c}$$

$$(\boldsymbol{A}_2 - \boldsymbol{A}_1^2)\cdot\boldsymbol{A}_1 = \dot{\boldsymbol{A}}_1\cdot\boldsymbol{A}_1 = \frac{\dot{\delta}_1}{2} \tag{3.8d}$$

and

$$\begin{aligned}\boldsymbol{A}_2\boldsymbol{m}\cdot\boldsymbol{A}_1\boldsymbol{m} &= \dot{\boldsymbol{A}}_1\boldsymbol{m}\cdot\boldsymbol{A}_1\boldsymbol{m} + (\boldsymbol{A}_1\boldsymbol{W} - \boldsymbol{W}\boldsymbol{A}_1)\boldsymbol{m}\cdot\boldsymbol{A}_1\boldsymbol{m} + \boldsymbol{A}_1^3\boldsymbol{m}\cdot\boldsymbol{m}\\ &= \frac{1}{2}[\dot{\delta}_4 - \boldsymbol{A}_1^3\boldsymbol{m}\cdot\boldsymbol{m} - (\boldsymbol{A}_1^2\boldsymbol{W}\boldsymbol{m} - \boldsymbol{W}\boldsymbol{A}_1^2\boldsymbol{m})\cdot\boldsymbol{m}] + \boldsymbol{A}_1^2\boldsymbol{W}\boldsymbol{m}\cdot\boldsymbol{m} + \boldsymbol{A}_1^3\boldsymbol{m}\cdot\boldsymbol{m}\\ &= \frac{1}{2}[\dot{\delta}_4 + \boldsymbol{A}_1^3\boldsymbol{m}\cdot\boldsymbol{m} + \underbrace{\boldsymbol{W}\cdot(\boldsymbol{A}_1^2\boldsymbol{m}\otimes\boldsymbol{m} + \boldsymbol{m}\otimes\boldsymbol{A}_1^2\boldsymbol{m})}_{=0}].\end{aligned} \tag{3.8e}$$

Then, by means of (2.4), (3.8) and on taking $\phi_0 = \phi_0^{(0)} + (\boldsymbol{A}_2\boldsymbol{m}\cdot\boldsymbol{m})\phi_0^{(1)}$, we deduce that the power per unit volume of the internal actions due to material dispersion reads

$$\begin{aligned}\frac{\boldsymbol{T}_D\cdot\boldsymbol{D}}{\rho} &= \frac{\phi_0^{(0)}}{2}\boldsymbol{A}_1\boldsymbol{m}\cdot\boldsymbol{m} + \frac{\phi_0^{(1)}}{2}(\boldsymbol{A}_2\boldsymbol{m}\cdot\boldsymbol{m})\boldsymbol{A}_1\boldsymbol{m}\cdot\boldsymbol{m} + \frac{\phi_1}{2}|\boldsymbol{A}_1|^2 + \frac{1}{2}(\phi_2\boldsymbol{A}_2 + \phi_3\boldsymbol{A}_1^2)\cdot\boldsymbol{A}_1\\ &\quad + \phi_4|\boldsymbol{A}_1\boldsymbol{m}|^2 + (\phi_5\boldsymbol{A}_2\boldsymbol{m} + \phi_6\boldsymbol{A}_1^2\boldsymbol{m})\cdot\boldsymbol{A}_1\boldsymbol{m}\\ &= \frac{\phi_0^{(0)}}{2}\dot{I}_4 + \frac{\phi_0^{(1)}}{4}\frac{\mathrm{D}\delta_2^2}{\mathrm{D}t} + \frac{\phi_2}{4}\dot{\delta}_1 + \frac{\phi_5}{2}\dot{\delta}_4\\ &\quad + \frac{1}{2}(\phi_2 + \phi_3)\mathrm{tr}(\boldsymbol{A}_1^3) + \left(\frac{\phi_5}{2} + \phi_6\right)\boldsymbol{A}_1^3\boldsymbol{m}\cdot\boldsymbol{m} + \frac{\phi_1}{2}|\boldsymbol{A}_1|^2 + \phi_4|\boldsymbol{A}_1\boldsymbol{m}|^2.\end{aligned} \tag{3.9}$$

Therefore, if

$$\phi_1 = \phi_4 = 0, \quad \phi_3 = -\phi_2, \quad \phi_6 = -\frac{\phi_5}{2}, \tag{3.10}$$

$\phi_0^{(0)}$, $\phi_0^{(1)}$, ϕ_2 and ϕ_5 depend on the reduced set of invariants $\{I_4, \delta_1, \delta_2^2, \delta_4\}$, and

$$\left.\begin{aligned} &\frac{\partial \phi_0^{(0)}}{\partial(\delta_2^2)} = \frac{1}{2}\frac{\partial \phi_0^{(1)}}{\partial I_4}, \quad \frac{\partial \phi_0^{(0)}}{\partial \delta_1} = \frac{1}{2}\frac{\partial \phi_2}{\partial I_4}, \quad \frac{\partial \phi_0^{(0)}}{\partial \delta_4} = \frac{\partial \phi_5}{\partial I_4}, \\ &\frac{\partial \phi_0^{(1)}}{\partial \delta_1} = \frac{\partial \phi_2}{\partial(\delta_2^2)}, \quad \frac{\partial \phi_0^{(1)}}{\partial \delta_4} = 2\frac{\partial \phi_5}{\partial(\delta_2^2)}, \quad \frac{\partial \phi_2}{\partial \delta_4} = 2\frac{\partial \phi_5}{\partial \delta_1}, \end{aligned}\right\} \tag{3.11}$$

then there exists a specific energy density

$$e_D = \hat{e}_D(I_4, \delta_1, \delta_2^2, \delta_4) \tag{3.12}$$

such that

$$\frac{\phi_0^{(0)}}{2} = \frac{\partial \hat{e}_D}{\partial I_4}, \quad \frac{\phi_0^{(1)}}{4} = \frac{\partial \hat{e}_D}{\partial(\delta_2^2)}, \quad \frac{\phi_2}{4} = \frac{\partial \hat{e}_D}{\partial \delta_1}, \quad \frac{\phi_5}{2} = \frac{\partial \hat{e}_D}{\partial \delta_4}, \tag{3.13}$$

and $\boldsymbol{T}_D \cdot \boldsymbol{D} = \rho \dot{e}_D$.

Finally, from (3.5), (3.10) and (3.13), we conclude that the part of the stress tensor due to material dispersion reads

$$\begin{aligned} \boldsymbol{T}_D = 2\rho &\left[\frac{\partial \hat{e}_D}{\partial I_4} + 2\frac{\partial \hat{e}_D}{\partial(\delta_2^2)}(\boldsymbol{A}_2\boldsymbol{m}\cdot\boldsymbol{m})\right]\boldsymbol{m}\otimes\boldsymbol{m} \\ &+ 4\rho\frac{\partial \hat{e}_D}{\partial \delta_1}\left(\boldsymbol{A}_2 - \boldsymbol{A}_1^2\right) + 2\rho\frac{\partial \hat{e}_D}{\partial \delta_4}\left[\left(\boldsymbol{A}_2 - \frac{1}{2}\boldsymbol{A}_1^2\right)\boldsymbol{m}\otimes\boldsymbol{m} + \boldsymbol{m}\otimes\left(\boldsymbol{A}_2 - \frac{1}{2}\boldsymbol{A}_1^2\right)\boldsymbol{m}\right]. \end{aligned} \tag{3.14}$$

As the specific energy density (3.12) is concerned, from (3.7) and the first equation in (3.13), we deduce that the response function $\hat{e}_D$ must satisfy the requirement

$$\frac{\partial \hat{e}_D}{\partial I_4}(I_4, 0, 0, 0) = 0, \tag{3.15}$$

for any $I_4 > 0$. In addition, due to thermodynamic restrictions [27], $\hat{e}_D$ must attain a minimum at equilibrium. This means that $\hat{e}_D$ has to be convex in a neighbourhood of $\dot{\boldsymbol{F}} = \boldsymbol{0}$. To satisfy this requirement, we impose that $\hat{e}_D$ satisfies the following inequalities:

$$\left.\begin{aligned} &\frac{\partial \hat{e}_D}{\partial \delta_1}(I_4, 0, 0, 0) > 0, \\ &2\frac{\partial \hat{e}_D}{\partial \delta_1}(I_4, 0, 0, 0) + \frac{\partial \hat{e}_D}{\partial \delta_4}(I_4, 0, 0, 0) > 0 \\ \text{and} \quad &\frac{\partial \hat{e}_D}{\partial \delta_1}(I_4, 0, 0, 0) + \frac{\partial \hat{e}_D}{\partial(\delta_2^2)}(I_4, 0, 0, 0) + \frac{\partial \hat{e}_D}{\partial \delta_4}(I_4, 0, 0, 0) > 0 \end{aligned}\right\} \tag{3.16}$$

for any $I_4 > 0$. The three positive quantities in (3.16) have physical dimensions $[L^2]$. In terms of them, three distinct characteristic lengths in a transversely isotropic material will be defined in the next section.

4. Infinitesimal theory

To properly derive the linearized theory of transversely isotropic dispersive materials, we introduce the displacement gradient $\boldsymbol{H} = \boldsymbol{F} - \boldsymbol{I}$ and assume that the strain is very small in the sense that

$$\varpi = \max_{(\boldsymbol{X},t)\in\mathcal{B}_r\times[0,T)} |\boldsymbol{H}(\boldsymbol{X}, t)| \ll 1. \tag{4.1}$$

After introducing the infinitesimal strain tensor $\varepsilon = (\boldsymbol{H} + \boldsymbol{H}^T)/2$ and the infinitesimal rate of strain tensor $\dot{\varepsilon}$, it is easy to verify that for an incompressible material, the following

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approximations hold:

$$I_1 \approx I_2 \approx 3 + 2\mathrm{tr}\varepsilon^2 + O(\varpi^3), \quad I_4 \approx 1 + 2\varepsilon \boldsymbol{M}\cdot\boldsymbol{M} + |\boldsymbol{HM}|^2 + O(\varpi^3), \tag{4.2a}$$

$$I_5 \approx 1 + 4\varepsilon \boldsymbol{M}\cdot\boldsymbol{M} + 4|\varepsilon \boldsymbol{M}|^2 + 2|\boldsymbol{HM}|^2 + O(\varpi^3), \tag{4.2b}$$

$$\delta_1 \approx 4\mathrm{tr}\dot{\varepsilon}^2 + O(\varpi^3), \quad \delta_2^2 \approx 4(\dot{\varepsilon}\boldsymbol{M}\cdot\boldsymbol{M})^2 + O(\varpi^3),\ \delta_4 \approx 4\dot{\varepsilon}^2\boldsymbol{M}\cdot\boldsymbol{M} + O(\varpi^3), \tag{4.2c}$$

$$\boldsymbol{B} \approx \boldsymbol{I} + 2\varepsilon + O(\varpi^2), \quad \boldsymbol{B}^{-1} \approx \boldsymbol{I} - 2\varepsilon + O(\varpi^2), \tag{4.2d}$$

$$\boldsymbol{m}\otimes\boldsymbol{m} \approx \boldsymbol{M}\otimes\boldsymbol{M} + \boldsymbol{HM}\otimes\boldsymbol{M} + \boldsymbol{M}\otimes\boldsymbol{HM} + O(\varpi^2), \tag{4.2e}$$

$$\boldsymbol{Bm}\otimes\boldsymbol{m} + \boldsymbol{m}\otimes\boldsymbol{Bm} \approx 2\boldsymbol{M}\otimes\boldsymbol{M} + 2(\varepsilon\boldsymbol{M}\otimes\boldsymbol{M} + \boldsymbol{M}\otimes\varepsilon\boldsymbol{M}) + 2(\boldsymbol{HM}\otimes\boldsymbol{M} + \boldsymbol{M}\otimes\boldsymbol{HM}) + O(\varpi^2) \tag{4.2f}$$

and

$$\boldsymbol{A}_1 \approx 2\dot{\varepsilon} + O(\varpi^2), \quad \boldsymbol{A}_2 \approx 2\ddot{\varepsilon} + O(\varpi^2),\ \boldsymbol{A}_2\boldsymbol{m}\cdot\boldsymbol{m} \approx 2\ddot{\varepsilon}\boldsymbol{M}\cdot\boldsymbol{M} + O(\varpi^2). \tag{4.2g}$$

Maclaurin expansions in the small parameter ϖ of W and e_D, combined with (3.3), (3.4), (3.15) and (4.2), yield (to leading order) the quadratic strain and internal energy densities

$$W \approx W^{(2)} = \mu_T \mathrm{tr}\,\varepsilon^2 + 2(\mu_L - \mu_T)\varepsilon^2\boldsymbol{M}\cdot\boldsymbol{M} + \frac{\beta}{2}(\varepsilon\boldsymbol{M}\cdot\boldsymbol{M})^2, \tag{4.3}$$

with

$$\mu_T = 2[\tilde{W}_1(3,3,1,1) + \tilde{W}_2(3,3,1,1)], \quad \mu_L = \mu_T + \tilde{W}_5(3,3,1,1),$$
$$\beta = 4[\tilde{W}_{44}(3,3,1,1) + 4\tilde{W}_{45}(3,3,1,1) + 4\tilde{W}_{55}(3,3,1,1)] \tag{4.4}$$

and

$$e_D \approx e_D^{(2)} = \upsilon_T \mathrm{tr}\,\dot{\varepsilon}^2 + 2(\upsilon_L - \upsilon_T)\dot{\varepsilon}^2\boldsymbol{M}\cdot\boldsymbol{M} + \frac{\gamma}{2}(\dot{\varepsilon}\boldsymbol{M}\cdot\boldsymbol{M})^2, \tag{4.5}$$

where

$$\upsilon_T = 4\frac{\partial\hat{e}_D}{\partial\delta_1}(1,0,0,0), \quad \upsilon_L = \upsilon_T + 2\frac{\partial\hat{e}_D}{\partial\delta_4}(1,0,0,0), \quad \gamma = 8\frac{\partial\hat{e}_D}{\partial(\delta_2^2)}(1,0,0,0). \tag{4.6}$$

In view of the convexity of (3.12) at equilibrium, the quadratic internal energy density $e_D^{(2)}$ is convex and hence the constants in (4.5) must satisfy the inequalities

$$\upsilon_L > 0, \quad \upsilon_T > 0, \quad \frac{\gamma}{2} + 2\upsilon_L - \upsilon_T > 0. \tag{4.7}$$

It is easy to check that these three positive constants are proportional to the quantities in (3.16) evaluated when the body is at equilibrium and unstretched in the preferred direction. Assuming that transversely isotropic material is a composite consisting of a hyperelastic ground matrix reinforced by a single family of fibres oriented along $\boldsymbol{M}$, the constants υ_L and υ_T are dispersion material parameters of the ground matrix in directions parallel and normal to $\boldsymbol{M}$, respectively, and γ provides a measure of the material dispersion in the fibres. In addition, the quantities $\sqrt{\upsilon_L}$ and $\sqrt{\upsilon_T}$ represent characteristic lengths in the ground matrix in directions parallel and normal to the preferred direction, respectively, and $\sqrt{\gamma/2 + 2\upsilon_L - \upsilon_T}$ is the characteristic length for the family of fibres.

Similarly, on assuming that the reference configuration is a local minimum of the strain energy density W, we have that

$$\mu_L > 0, \quad \mu_T > 0, \quad \frac{\beta}{2} + 2\mu_L > \mu_T. \tag{4.8}$$

The constants μ_L and μ_T are the stiffnesses of the ground matrix in directions parallel and normal to $\boldsymbol{M}$, respectively, and β gives a measure of the stiffness of the fibres.

To leading order, the Maclaurin expansion of the stress Cauchy stress tensor $\boldsymbol{T} = -p\boldsymbol{I} + \boldsymbol{T}_E^{(a)} + \boldsymbol{T}_D$, with $\boldsymbol{T}_E^{(a)}$ as in (3.2) and $\boldsymbol{T}_D$ as in (3.14), yields

$$\begin{aligned}\boldsymbol{T} \approx \boldsymbol{\sigma} = -p\boldsymbol{I} + \beta(\varepsilon \boldsymbol{M}\cdot\boldsymbol{M})\boldsymbol{M}\otimes\boldsymbol{M} + 2\mu_T\varepsilon + 2(\mu_L-\mu_T)(\varepsilon\boldsymbol{M}\otimes\boldsymbol{M} + \boldsymbol{M}\otimes\varepsilon\boldsymbol{M})\\ + \rho\gamma(\ddot{\varepsilon}\boldsymbol{M}\cdot\boldsymbol{M})\boldsymbol{M}\otimes\boldsymbol{M} + 2\rho\upsilon_T\ddot{\varepsilon} + 2\rho(\upsilon_L-\upsilon_T)(\ddot{\varepsilon}\boldsymbol{M}\otimes\boldsymbol{M} + \boldsymbol{M}\otimes\ddot{\varepsilon}\boldsymbol{M}).\end{aligned} \tag{4.9}$$

Observe that, if $\mu_L = \mu_T$ and $\beta = 0$ (i.e. the material has equal stiffness in all directions), and $\upsilon_L = \upsilon_T$ and $\gamma = 0$ (i.e. the three characteristic lengths coincide), then (4.9) gives the stress tensor of an isotropic dispersive material in the linear regime.

Finally, we observe that at small strains and rates of strain, the first Piola–Kirchhoff and the Cauchy stress tensors coincide, i.e. $\boldsymbol{P} \approx \boldsymbol{T} \approx \boldsymbol{\sigma}$ and the linearized equation of motion reads

$$\rho\frac{\partial^2\boldsymbol{x}}{\partial t^2} = \mathrm{Div}\,\boldsymbol{\sigma}. \tag{4.10}$$

(a) Anti-plane shear motions

Let us consider the anti-plane shear motion given by

$$x_1 = X_1, \quad x_2 = X_2, \quad x_3 = X_3 + u_3(X_1, X_2, t). \tag{4.11}$$

It can be proved that, in the linear regime, a transversely isotropic material can undergo this class of motions providing that the preferred direction $\boldsymbol{M}$ lies in the X_1X_2 plane. Then, on taking $\boldsymbol{M} = M_1\boldsymbol{e}_1 + M_2\boldsymbol{e}_2$ (with $M_1^2 + M_2^2 = 1$), from (4.10), the equation for the unknown u_3 is found to be

$$\begin{aligned}\rho\frac{\partial^2 u_3}{\partial t^2} &= [(\mu_L-\mu_T)M_1^2 + \mu_T]\frac{\partial^2 u_3}{\partial X_1^2} + 2(\mu_L-\mu_T)M_1M_2\frac{\partial^2 u_3}{\partial X_1\partial X_2}\\ &+ [(\mu_L-\mu_T)M_2^2 + \mu_T]\frac{\partial^2 u_3}{\partial X_2^2} + \rho[(\upsilon_L-\upsilon_T)M_1^2 + \upsilon_T]\frac{\partial^4 u_3}{\partial X_1^2\partial t^2}\\ &+ 2\rho(\upsilon_L-\upsilon_T)M_1M_2\frac{\partial^4 u_3}{\partial X_1\partial X_2\partial t^2} + \rho[(\upsilon_L-\upsilon_T)M_2^2 + \upsilon_T]\frac{\partial^4 u_3}{\partial X_2^2\partial t^2}.\end{aligned} \tag{4.12}$$

We now look for wave solutions of (4.11) in the form

$$u_3(X_1, X_2, t) = A\sin[\omega t - k(n_1X_1 + n_2X_2)], \tag{4.13}$$

where the positive constants A, ω and k represent, respectively, the amplitude, the angular frequency and the wavenumber, and n_1 and n_2, with $n_1^2 + n_2^2 = 1$, are the components of the direction of propagation $\boldsymbol{n}$. Inserting (4.13) into (4.12), we deduce that progressive sinusoidal waves can propagate in a transversely isotropic dispersive solid providing that the following dispersion relation is satisfied:

$$\frac{\omega^2}{k^2} = \frac{(\boldsymbol{M}\cdot\boldsymbol{n})^2(c_L^2 - c_T^2) + c_T^2}{1 + k^2[(\boldsymbol{M}\cdot\boldsymbol{n})^2(\upsilon_L - \upsilon_T) + \upsilon_T]}, \tag{4.14}$$

where $c_T = \sqrt{\mu_T/\rho}$ and $c_L = \sqrt{\mu_L}\rho$ are the speeds at which transverse waves would propagate in a direction perpendicular or parallel to the preferred direction in the absence of material dispersion.

Equation (4.14) tells us that only waves with angular frequencies in the range $[0, \omega_n^\star[$ with

$$\omega_n^\star = \sqrt{\frac{(\boldsymbol{M}\cdot\boldsymbol{n})^2(c_L^2 - c_T^2) + c_T^2}{(\boldsymbol{M}\cdot\boldsymbol{n})^2(\upsilon_L - \upsilon_T) + \upsilon_T}} \tag{4.15}$$

can propagate, and the group velocity $v_n = \omega/k$ depends on the angle contained between the preferred direction and the direction of propagation (cf. (4.14)). The minimum and maximum

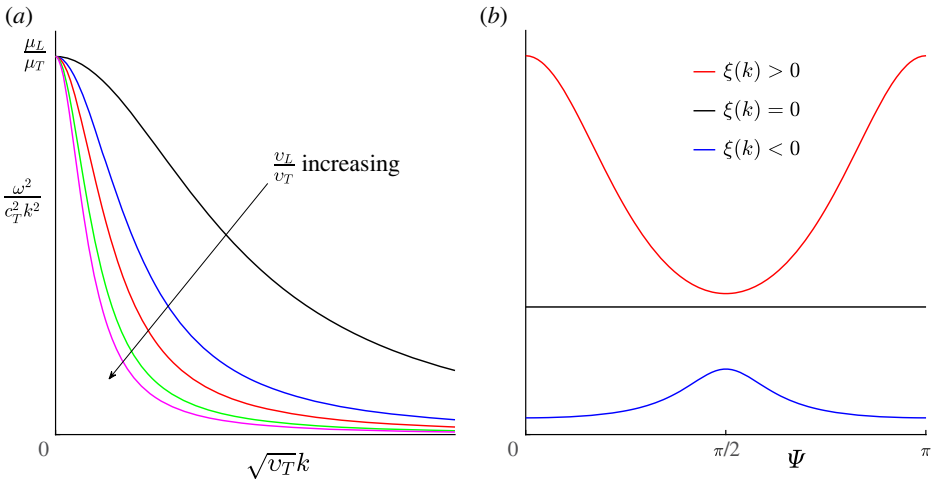

Figure 1. The dimensionless group velocity $\omega^2/(c_T^2 k^2)$ is plotted as a function of (*a*) the dimensionless wavenumber $\sqrt{v_T}k$ for different values of the ratio v_L/v_T and $\Psi = 0$ and (*b*) the convex angle Ψ for different values of k.

values of the group velocity,

$$v_{\min} = \min_{\boldsymbol{n}\in S^1} v_{\boldsymbol{n}} \quad \text{and} \quad v_{\max} = \max_{\boldsymbol{n}\in S^1} v_{\boldsymbol{n}}, \tag{4.16}$$

are attained when the direction of propagation of the wave is parallel or perpendicular to the preferred direction. More precisely, introducing the wavenumber-dependent function

$$\xi(k) = k^2(\mu_L v_T - \mu_T v_L) + \mu_L - \mu_T, \tag{4.17}$$

and the group velocities of the waves propagating parallel and perpendicularly to the preferred direction

$$v_{\parallel} = \frac{c_L}{\sqrt{1 + k^2 v_L}} \quad \text{and} \quad v_{\perp} = \frac{c_T}{\sqrt{1 + k^2 v_T}}, \tag{4.18}$$

by minimizing/maximizing (4.14) with respect to the convex angle Ψ contained between $\boldsymbol{M}$ and $\boldsymbol{n}$, we deduce that

- if $\xi(k) > 0$, then $v_{\max} = v_{\parallel}$ and $v_{\min} = v_{\perp}$;
- if $\xi(k) < 0$, then $v_{\max} = v_{\perp}$ and $v_{\min} = v_{\parallel}$;
- if $\xi(k) = 0$ (this is the case only when $\kappa^\star = (\mu_L - \mu_T)/(\mu_T v_L - \mu_L v_T) > 0$ and $k = \sqrt{\kappa^\star}$), then the group velocity is equal to

$$v^\star = \sqrt{\frac{c_T^2 v_L - c_L^2 v_T}{v_L - v_T}} \tag{4.19}$$

irrespective of the angle Ψ.

Figure 1*a* displays the dimensionless group velocity $\omega^2/(c_T^2 k^2)$ versus the dimensionless wavenumber $\sqrt{v_T}k$ at different values of the ratio v_L/v_T and for anti-plane shear waves propagating in the preferred direction (i.e. $\Psi = 0$). Figure 1*b* shows instead the dependence of the dimensionless group velocity on the angle contained between the preferred direction and the direction of propagation of the shear wave for positive, vanishing and negative values of the wavenumber-dependent function ξ (cf. (4.17)).

5. Solitary wave solutions

In this section, we focus our attention on shearing motions in the form

$$x_1 = X_1 + u(X_2, t), \quad x_2 = X_2, \; x_3 = X_3 \tag{5.1}$$

that a transversely isotropic dispersive body can undergo in the case in which the preferred direction lies in the X_1X_2 plane and is parametrized as $\boldsymbol{M} = \cos\Theta \boldsymbol{e}_1 + \sin\Theta \boldsymbol{e}_2$. Since the directions $\boldsymbol{M}$ and $-\boldsymbol{M}$ are physically indistinguishable, the angle Θ is assumed to belong to the interval $[0, \pi[$. In view of the analytical form of this class of motions, from (2.8), (2.9), (3.2) and (3.14), we deduce that the Lagrange multiplier p depends only on the spatial variable X_2 as well as on time. In addition, we realize that it is convenient to write the scalar equations of motion in terms of the transverse strain $U = \dfrac{\partial u}{\partial X_2}$. In so doing, the equations for U and p read

$$\rho \frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 P_{12}}{\partial X_2^2} \tag{5.2a}$$

and

$$\frac{\partial P_{22}}{\partial X_2} = 0, \tag{5.2b}$$

where the components P_{12} and P_{22} of the Piola–Kirchhoff stress tensors are as follows:

$$\begin{aligned} P_{12} = {} & 2(\tilde{W}_1 + \tilde{W}_2)U + 2\tilde{W}_4 \sin\Theta(\cos\Theta + U\sin\Theta) \\ & + 2\tilde{W}_5[2\sin\Theta\cos\theta + (1 + 2\sin^2\Theta)U + 3U^2\sin\Theta\cos\Theta + 2U^3\sin^2\Theta] \\ & + 2\rho\frac{\partial e_D}{\partial I_4}\sin\Theta(\cos\Theta + U\sin\Theta) \\ & + 8\rho\frac{\partial \hat{e}_D}{\partial(\delta_2^2)}\sin^2\Theta(\cos\Theta + U\sin\Theta)\left[\frac{\partial^2 U}{\partial t^2}(\cos\Theta + U\sin\Theta) + \left(\frac{\partial U}{\partial t}\right)^2 \sin\Theta\right] + 4\rho\frac{\partial \hat{e}_D}{\partial \delta_1}\frac{\partial^2 U}{\partial t^2} \\ & + 2\rho\frac{\partial \hat{e}_D}{\partial \delta_4}\left\{\frac{\partial^2 U}{\partial t^2}[1 + U\sin\Theta(2\cos\Theta + U\sin\Theta)] + \left(\frac{\partial U}{\partial t}\right)^2 \sin\Theta(\cos\Theta + U\sin\Theta)\right\} \end{aligned} \tag{5.2c}$$

and

$$\begin{aligned} P_{22} = {} & -p + 2\tilde{W}_1 - 2\tilde{W}_2(1 + U^2) + 2\tilde{W}_4\sin^2\Theta + 4\tilde{W}_5\sin\Theta[\sin\Theta + U(\cos\Theta + U\sin\Theta)] \\ & + 2\rho\frac{\partial \hat{e}_D}{\partial I_4}\sin^2\Theta + 8\rho\frac{\partial \hat{e}_D}{\partial(\delta_2^2)}\sin^3\Theta\left[\frac{\partial^2 U}{\partial t^2}(\cos\Theta + U\sin\Theta) + \left(\frac{\partial U}{\partial t}\right)^2 \sin\Theta\right] \\ & + 6\rho\frac{\partial \hat{e}_D}{\partial \delta_1}\left(\frac{\partial U}{\partial t}\right)^2 \\ & + 2\rho\frac{\partial \hat{e}_D}{\partial \delta_4}\sin\Theta\left[\frac{\partial^2 U}{\partial t^2}(\cos\Theta + U\sin\Theta) + 3\left(\frac{\partial U}{\partial t}\right)^2 \sin\Theta\right]. \end{aligned} \tag{5.2d}$$

The invariants on which the derivatives of the strain and internal energy densities in (5.2*c*) and (5.2*d*) depend are given by

$$I_1 = I_2 = 3 + U^2, \tag{5.2e}$$

$$I_4 = 1 + U\sin\Theta(2\cos\Theta + U\sin\Theta) \equiv \mathcal{I}_4(U;\Theta), \tag{5.2f}$$

$$I_5 = 1 + 4U\sin\Theta\cos\Theta + U^2(1 + 2\sin^2\Theta) + 2U^3\sin\Theta\cos\Theta + U^4\sin^2\Theta \equiv \mathcal{I}_5(U;\Theta), \tag{5.2g}$$

$$\delta_1 = 2\left(\frac{\partial U}{\partial t}\right)^2, \tag{5.2h}$$

$$\delta_2^2 = 4\left(\frac{\partial U}{\partial t}\right)^2 \sin^2\Theta(\cos\Theta + U\sin\Theta)^2 \equiv \Delta_2\left(U, \left(\frac{\partial U}{\partial t}\right)^2; \Theta\right) \tag{5.2i}$$

and

$$\delta_4 = \left(\frac{\partial U}{\partial t}\right)^2 [1 + U\sin\Theta(2\cos\Theta + U\sin\Theta)] \equiv \Delta_4\left(U, \left(\frac{\partial U}{\partial t}\right)^2; \Theta\right). \tag{5.2j}$$

Equations (5.2*a*) and (5.2*c*) give the nonlinear Love equation in a transversely isotropic dispersive material. Once this equation has been solved, the Lagrange multiplier p can be easily determined from (5.2*b*) and (5.2*d*). For this reason, we focus our analysis only on equation (5.2*a*). In particular, we look for travelling wave solutions in the form

$$U = \widehat{U}(s), \quad \text{with } s = X_2 - ct, \tag{5.3}$$

and c being the speed at which the wave propagates in the X_2-direction, such that

$$\lim_{s\to\pm\infty} \widehat{U}(s) = 0. \tag{5.4}$$

Inserting the ansatz (5.3) into (5.2*a*) and (5.2*c*), integrating twice the resulting equation by taking into account the conditions at infinity (5.4) one arrives at the first integral

$$\frac{\rho}{2}c^2\widehat{U}^2 = \mathcal{W}(\widehat{U};\Theta) + \rho c^2\mathcal{E}_D(\widehat{U}, \widehat{U}'^2;\Theta), \tag{5.5a}$$

where

$$\mathcal{W}(\widehat{U};\Theta) = \hat{W}(3 + \widehat{U}^2, 3 + \widehat{U}^2, \mathcal{I}_4(\widehat{U};\Theta), \mathcal{I}_5(\widehat{U},\Theta)) \tag{5.5b}$$

and

$$\mathcal{E}_D(\widehat{U}, \widehat{U}'^2;\Theta) = \hat{e}_D(\mathcal{I}_4(\widehat{U};\Theta), 2c^2\widehat{U}'^2, \Delta_2(\widehat{U}, c^2\widehat{U}'^2;\Theta), \Delta_4(\widehat{U}, c^2\widehat{U}'^2;\Theta)). \tag{5.5c}$$

If $\partial\mathcal{E}_D/\partial(\widehat{U}'^2) \neq 0$, then (5.5*a*) can be locally solved for $\widehat{U}'^2$ to give

$$\widehat{U}'^2 = \mathcal{G}(\widehat{U};\Theta). \tag{5.6}$$

An analysis *à la* Weierstrass reveals that if $\widehat{U} = 0$ is a zero of $\mathcal{G}$ with multiplicity greater than two and there exists $\widehat{U}_M > 0$ (respectively, $\widehat{U}_m < 0$), a zero of $\mathcal{G}$ with multiplicity less than or equal to one such that $\mathcal{G} > 0$ in $]0, \widehat{U}_M[$ (respectively, in $]U_m, 0[$), then there exist solutions of (5.6) that satisfy the conditions at infinity (5.4). These solutions represent bright solitary waves in the case in which $\mathcal{G}$ is positive in $]0, U_M[$, dark solitary waves when $\mathcal{G}$ is positive in $]U_m, 0[$.

For illustration, we consider the following models for the strain and internal energy densities:

$$W = \frac{\mu}{2}\left[(I_1 - 3) + \frac{\varsigma_{nl}}{2}(I_1 - 3)^2 + \frac{\varsigma}{2}(I_4 - 1)^2\right] \tag{5.7}$$

and

$$e_D = \frac{\upsilon_1}{4}(\boldsymbol{A}_1\boldsymbol{m}\cdot\boldsymbol{m})^2 + \frac{\upsilon_2}{4}\mathrm{tr}\boldsymbol{A}_1^2 + \upsilon_3(\boldsymbol{A}_1^2\boldsymbol{m}\cdot\boldsymbol{m}). \tag{5.8}$$

The response function (5.7) gives the strain energy density of an isotropic elastic matrix, with infinitesimal shear modulus $\mu > 0$, reinforced by a family of fibres whose orientation originates the preferred direction. The dimensionless constants ς_{nl} and ς are taken positive so that the elastic potential attains the absolute minimum at the undeformed reference configuration. Similarly, the

constants υ's in (5.8) are assumed to be positive in order that the specific internal energy is a minimum in equilibrium (cf. (3.16)). With these constitutive models, equation (5.6) reads

$$\widehat{U}'^2 = \frac{2(c^2 - c_T^2) - c_T^2 \varsigma_{nl} \widehat{U}^2 - c_T^2 \varsigma \sin^2\Theta (4\cos^2\Theta + 4\widehat{U}\sin\Theta\cos\Theta + \widehat{U}^2\sin^2\Theta)}{2c^2[\upsilon_2 + 2\upsilon_3 \sin^2\Theta + 2(\upsilon_1 \sin^2\Theta + \upsilon_3)(\cos\Theta + \widehat{U}\sin\Theta)^2]}\widehat{U}^2, \tag{5.9}$$

where $c_T = \sqrt{\mu/\rho}$ represents now the speed of transverse waves in the isotropic matrix in the linear regime. On the basis of the results of the analysis *à la* Weierstrass reported above and by studying the positiveness of the right-hand side of equation (5.9), we deduce that equation (5.9) admits solitary wave solutions if and only if

$$(a)\quad c^2 > c_T^2(1 + 2\varsigma \sin^2\Theta \cos^2\Theta) = c_\star^2(\Theta) \quad \text{or} \quad (b)\quad c^2 = c_\star^2(\theta) \quad \text{and} \quad \Theta \in]0, \pi\left[- \left\{\frac{\pi}{2}\right\}\right. . \tag{5.10}$$

In particular, in case (*a*) equation (5.9) admits both bright and dark solitary wave solutions; in case (*b*) equation (5.9) admits only dark solitary wave if $\Theta \in]0, \pi/2[$, only bright solitary waves if $\Theta \in]\pi/2, \pi[$.

For brevity, we solve (5.9) only for $c^2 > c_\star^2$ and when the preferred direction is orthogonal (i.e. $\Theta = 0$) or parallel (i.e. $\Theta = \pi/2$) to the direction of propagation of the wave. Specifically, for $\Theta = 0$ and $c^2 > c_\star^2(0) = c_T^2$,

$$\widehat{U}_b^\perp(s) = \sqrt{\frac{2(c^2 - c_T^2)}{c_T^2 \varsigma_{nl}}}\operatorname{sech}\left[\sqrt{\frac{c^2 - c_T^2}{c^2(\upsilon_2 + 2\upsilon_3)}}(s - s_0)\right] \quad \text{and} \quad \widehat{U}_d^\perp(s) = -\widehat{U}_b^\perp(s), \tag{5.11}$$

with s_0 being an integration constant, are the bright and dark solitary wave solutions of (5.9). For $\Theta = \pi/2$ and $c^2 > c_\star^2(\pi/2) = c_T^2$, the bright and dark solitary wave solutions of (5.9) are given implicitly. More precisely, introducing the auxiliary function

$$\mathcal{F} : \xi \in]0, r^{-1}] \mapsto \frac{a}{r}\left[\arctan\left(\frac{r}{a}\xi\right) - \arctan\left(\frac{1}{a}\right)\right] - \operatorname{arctanh}\sqrt{\frac{1 - r^2\xi^2}{1 + a^2\xi^2}} \in]-\infty, 0], \tag{5.12}$$

with

$$a = \sqrt{\frac{2(\upsilon_1 + \upsilon_3)}{\upsilon_2 + 2\upsilon_3}} \quad \text{and} \quad r = \sqrt{\frac{c_T^2(\varsigma_{nl} + \varsigma)}{2(c^2 - c_T^2)}}, \tag{5.13}$$

the bright and dark solitary wave solutions are, respectively, given by

$$\widehat{U}_b^\parallel(s) = \begin{cases} \mathcal{F}^{-1}\left[\sqrt{\dfrac{c^2 - c_T^2}{c^2(\upsilon_2 + 2\upsilon_3)}}(s - s_0)\right] & \text{if } s - s_0 \le 0, \\ \mathcal{F}^{-1}\left[-\sqrt{\dfrac{c^2 - c_T^2}{c^2(\upsilon_2 + 2\upsilon_3)}}(s - s_0)\right] & \text{if } s - s_0 > 0, \end{cases} \tag{5.14}$$

and $\widehat{U}_d^\parallel(s) = -\widehat{U}_b^\parallel(s)$.

From (5.11), (5.12) and (5.14), we deduce that

$$\max|\widehat{U}_i^\parallel| = \sqrt{\frac{\varsigma_{nl}}{\varsigma_{nl} + \varsigma}}\max|\widehat{U}_i^\perp| < \max|\widehat{U}_i^\perp| \quad (i = b, d), \tag{5.15}$$

whence the maximum amplitude of the solitary waves propagating parallel to the direction of the fibres is less than that of the solitary waves propagating perpendicularly to the preferred direction (figure 2). Moreover, if $\upsilon_1 \gg \max\{\upsilon_2, \upsilon_3\}$, then $a \to +\infty$ so that the auxiliary function (5.12) tends

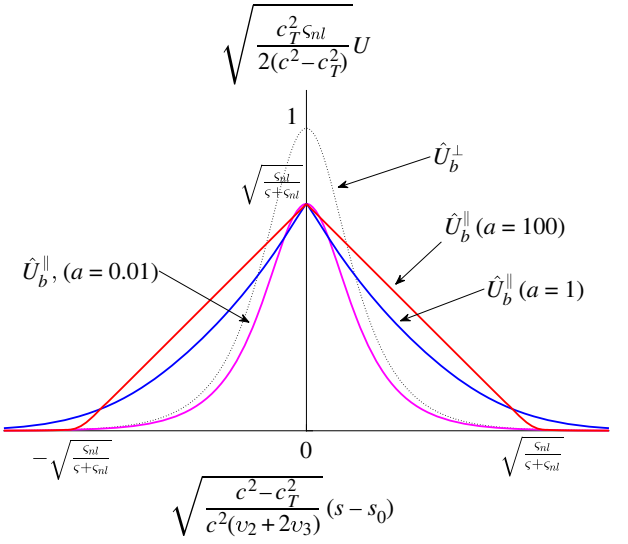

Figure 2. Bright solitary waves propagating perpendicularly (dotted black line) or parallel (coloured solid lines) to the direction of the fibres. For $\Theta=\pi/2$, the solitary wave $\hat{U}_b^{\parallel}$ tends to the wave with compact support $U_{b\,\infty}^{\parallel}$ (cf. (5.16)) in the limit as $\max\{\upsilon_2,\upsilon_3\}/\upsilon_1\to 0^+$. For $a=100$, the profiles of the bright solitary wave $\hat{U}_b^{\parallel}$ and the pulse $U_{b\,\infty}^{\parallel}$ are practically indistinguishable.

to $\mathcal{F}_\infty:\xi\in]0,r^{-1}]\mapsto\xi-r^{-1}\in]-r^{-1},0]$, and, consequently, the bright (respectively, dark) solitary wave (5.14) tends uniformly to the bright solitary pulse wave

$$U_{b\,\infty}^{\parallel}=\begin{cases}\sqrt{\dfrac{c^2-c_T^2}{c^2(\upsilon_2+2\upsilon_3)}}(s-s_0)+r^{-1} & \text{if } \sqrt{\dfrac{c^2-c_T^2}{c^2(\upsilon_2+2\upsilon_3)}}(s-s_0)\in\ ]-r^{-1},0],\\ -\sqrt{\dfrac{c^2-c_T^2}{c^2(\upsilon_2+2\upsilon_3)}}(s-s_0)+r^{-1} & \text{if } \sqrt{\dfrac{c^2-c_T^2}{c^2(\upsilon_2+2\upsilon_3)}}(s-s_0)\in\ ]0,r^{-1}[,\\ 0 & \text{if } \sqrt{\dfrac{c^2-c_T^2}{c^2(\upsilon_2+2\upsilon_3)}}|s-s_0|\geq r^{-1}\end{cases}\tag{5.16}$$

(respectively, to the dark solitary pulse wave $U_{d\,\infty}^{\parallel}=-U_{b\,\infty}^{\parallel}$). Figure 2 displays the profiles of the bright solitary waves (5.14) for different values of a. For great enough values of a, the wave solution (5.14) almost coincides with the pulse given by (5.16).

6. Concluding remarks

Inspired by Fosdick & Warner [30] in modelling dispersion within the theory of simple materials *à la* Noll, by Rubin *et al.* [16] in splitting the stress tensor into the sum of an elastic and a dispersive parts, with the dispersive part deriving from a potential that depends on both isotropic and anisotropic invariants of the first two Rivlin–Ericksen tensors, and by the results in [17] on the connection between the RRG theory and simple materials, we have introduced a constitutive model for material dispersion in a transversely isotropic hyperelastic material. In developing our model, we have used a series of results of Edelen [24–26] on the form of the constitutive relations consistent with the second law of thermodynamics. Even though Edelen had introduced the theoretical background necessary to develop a theory for dispersion in simple materials, it seems that he was not interested in using his findings to model material dispersion. Edelen was more interested in strain rate-dependent models that do not dissipate energy. Incidentally,

more recently, Rajagopal & Srinivasa [31] have proposed implicit constitutive equations for the mechanical response of a class of materials that do not dissipate mechanical work in any process.

Our theoretical framework allows a more realistic characterization of the mechanical behaviour of soft tissue by ultrasonic measurements. For example, by combining our results with the double frequency continuous wave method developed by Levy *et al.* [6] one could measure the speeds of sound dispersion in different directions. Ultrasound being a non-invasive diagnostic tool, another notable potential application of our theory could be the *in vivo* characterization of the collagen fibre microstructure of aortic walls. Thus far, the collagen microstructure has been examined by analysing *ex vivo* data from second-harmonic generation imaging after optical clearing [32].

We have derived the infinitesimal theory by which we have discerned three characteristic lengths: one in the preferred direction, and the other two on surfaces parallel to $\boldsymbol{M}$ in directions parallel and normal to $\boldsymbol{M}$. We have studied also the propagation of bulk anti-plane shear waves in materials with a preferred direction in the linear regime. In the special case in which the stiffnesses and the characteristic lengths of the material are equal in all directions, i.e. in the isotropic case, setting $\mu_L = \mu_T = \rho C^2$, with C having the same physical dimensions as the velocity, and realizing that $\upsilon_L = \upsilon_T = \alpha_1(0)$, we deduce that equation (4.12) reduces to

$$\frac{\partial^2 u_3}{\partial t^2} = \nabla^2 \left(C^2 u_3 + \alpha_1(0) \frac{\partial^2 u_3}{\partial t^2} \right), \tag{6.1}$$

where ∇^2 is the two-dimensional Laplace operator.

Except for a nonlinear term in the divergence of the elastic part of the stress tensor, eqn (3.12) in [8] coincides with (6.1) providing that

$$\alpha_1(0) = \frac{l^2}{12}, \tag{6.2}$$

where l is the equilibrium interparticle distance of a mass–spring chain. Specifically, Rosenau [8] derived his equation within the theory of dense lattices by using a continualization method based on the one-point Padé approximation of the differential operators (instead of the usual Taylor series expansion) and by dropping a couple of (linear) terms that account for material dispersion but are *not* invariant under infinitesimal rotations. Reporting his own words,

> the last two terms in equations (3.10) and (3.11)...are non-invariant under infinitesimal rotations. Since the first three terms of equation (3.11) have an invariant form, it seems advantageous to drop the invariance-breaking terms and to study

eqn (3.12). We have instead derived the linearized version of eqn (3.12) in [8] from a phenomenological continuum model of dispersion without the necessity of dropping *a posteriori* terms that break the material symmetry. This is the main advantage of having derived our model within a phenomenological theory instead of in the framework of dense lattices. However, it must be said that the modelling methodology based on a microscopic approach outlined in [8,33] may be adapted (by requiring the invariance of the lattice under any rotation about a given direction instead of rotations by integer multiple of π) to give a justification of our phenomenological continuum model from a discrete mechanical perspective. Suffice it to observe that, in an isotropic dispersive material, the inherent characteristic lengths envisaged by phenomenological and discrete models are related through (6.2).

Finally, we have studied the propagation of finite amplitude solitary waves in a fibre-reinforced dispersive material and pointed out that solitary waves with compact support, i.e. pulses, can propagate in the direction of the fibres.

Further research on the propagation of waves with compact support in anisotropic dispersive materials could generalize the results obtained for isotropic bodies in [17].

Data accessibility. This article does not contain any additional data.

Declaration of AI use. We have not used AI-assisted technologies in creating this article.

Authors' contributions. A.A.: formal analysis, investigation, writing—review and editing; J.d.C.M.: formal analysis, investigation, writing—review and editing; G.S.: conceptualization, investigation, writing—review and editing; L.V.: formal analysis, investigation, writing—original draft.

All authors gave final approval for publication and agreed to be held accountable for the work performed therein.

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